Dynamical analysis of the exclusive queueing process

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In a recent study [1] the stationary state of a parallel-update TASEP with varying system length, which can be regarded as a queueing process with excluded-volume effect (*exclusive queueing process*, EQP), was obtained. We analyze the dynamical properties of the number of particles $\langle N_t \rangle$ and the position of the last particle (the system length) $\langle L_t \rangle$, using an analytical method (generating function technique) as well as a phenomenological description based on domain wall dynamics and Monte Carlo simulations. The system exhibits two phases corresponding to linear convergence or divergence of $\langle N_t \rangle$ and $\langle L_t \rangle$. These phases can both further be subdivided into high-density and maximal-current subphases. The predictions of the domain wall theory are found to be in very good agreement quantitatively with results from Monte Carlo simulations in the convergent phase. On the other hand, in the divergent phase, only the prediction for $\langle N_t \rangle$ agrees with simulations.

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I. INTRODUCTION

Queueing processes have been studied extensively, especially due to their practical relevance [2–4]. However, usually the spatial structure of the queues is neglected and the particles in the queues do not interact with each other. On the other hand, the totally asymmetric simple exclusion process (TASEP) which has a spatial structure and excluded-volume effect (hard-core repulsion) is one of the best-studied interacting particle systems [5]. Nowadays the TASEP is a basic model for pedestrian and traffic flows [6, 7].

Recently a queueing process with the excluded-volume effect, the exclusive queueing process (EQP), was introduced in [8] and [9] independently, where the model was formulated as continuous-time and discrete-time Markov processes, respectively. This model can be rephrased as the TASEP on a semi-infinite chain with a new boundary condition, see Fig. 1. The left end is interpreted as the end of the queue where new customers arrive. Therefore particles can enter at the left site next to the leftmost occupied site. The (fixed) rightmost site corresponds to

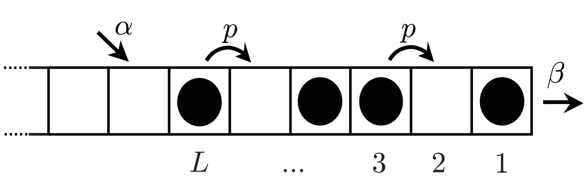

FIG. 1: Exclusive queueing process.

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the server. Here particles can leave the system after getting service. In the bulk a particle can hop to its right nearest neighbor site, if the target site is empty. After the study [9], where the bulk hopping rule is deterministic, the model with discrete time and probabilistic bulk hopping was analyzed [1].

Earlier works [1, 8, 9] focussed on the exact probability distribution, physical quantities in the stationary state, and the conditions under which the EQP converges to the stationary state. In this paper, we study the dynamical properties by considering the number of particles and the system length which is defined as the position of the leftmost particle. We use the same formalism as in [1, 9], i.e. discrete time and parallel-update scheme. For generic hopping probability p, we will introduce a domain wall prediction, checking it by Monte Carlo simulations. In the deterministic hopping case $p = 1$, a rigorous analysis is available, by using the generating function technique [13].

This paper is organized as follows. In Sec. II, we define the model as a discrete-time Markov process, and review its stationary state based on [1], which can be generalized to the inhomogeneous injection case, see also App. A. In Sec. III, we introduce a phenomenological argument on how the number of particles $\langle N_t \rangle$ and the system length $\langle L_t \rangle$ converge or diverge, showing simulation results. In Sec. IV, we derive the asymptotic behaviors of $\langle N_t \rangle$ and $\langle L_t \rangle$ rigorously for $p = 1$, imposing the initial condition that there is no particle on the chain. Section V is devoted to the conclusion of this paper.

II. MODEL

The EQP is defined on a semi-infinite chain where sites are labeled by natural numbers from right to left (Fig. 1). Particles can enter the chain with probability α only at the left site next to the leftmost occupied site. A particle

hops to its right nearest neighbor site with probability p, if it is empty, and exits at the right end of the chain with probability β. If there is no particle on the chain, a particle enters at site 1 with probability α. These transitions occur simultaneously within one time step, i.e. we apply the fully-parallel-update scheme.

We formulate the EQP as a discrete-time Markov process on the state space

$$S = \{\emptyset, 1, 10, 11, 100, 101, 110, 111, 1000, \dots\} \quad (1)$$

where 0 and 1 correspond to unoccupied and occupied sites, respectively. In particular, $\emptyset$ denotes the state in which there is no particle on the chain. To simplify the notation we do not write the infinite number of 0's located left to the leftmost 1.

Let us review the matrix product stationary state [1] of the model, which is a simple extension of that for systems with a fixed system length [10]. When

$$\begin{cases} \alpha \le \alpha_c = \frac{1-\sqrt{1-p}}{2} & \text{for } \beta > 1-\sqrt{1-p}, \\ \alpha < \alpha_c = \frac{\beta(p-\beta)}{p-\beta^2} & \text{for } \beta \le 1-\sqrt{1-p}, \end{cases} \quad (2)$$

the stationary state can be expressed as

$$P(\emptyset) = \frac{1}{Z}\,, \quad (3)$$

$$P(1\tau_{L-1}\dots\tau_1) = \frac{1}{Z}\left(\frac{\alpha}{p(1-\alpha)}\right)^L \langle W|DX_{\tau_{L-1}}\cdots X_{\tau_1}|V\rangle\,. \quad (4)$$

$X_1 = D$ and $X_0 = E$ are matrices, $\langle W|$ is a row vector and $|V\rangle$ is a column vector satisfying the algebraic relations

$$\begin{aligned}
EDEE =& (1-p)EDE + EEE + pEE, \\
EDED =& EDD + EED + pED, \\
DDEE =& (1-p)DDE + (1-p)DEE \\
& + p(1-p)DE, \\
DDED =& DDD + (1-p)DED + pDD, \\
DDE|V\rangle =& (1-\beta)DD|V\rangle + (1-p)DE|V\rangle \\
& + p(1-\beta)D|V\rangle, \\
EDE|V\rangle =& (1-\beta)ED|V\rangle + EE|V\rangle + pE|V\rangle, \\
\langle W|DEE =& (1-p)\langle W|DE, \\
\langle W|DED =& \langle W|DD + p\langle W|D, \\
DD|V\rangle =& \frac{p(1-\beta)}{\beta}D|V\rangle,\ ED|V\rangle = \frac{p}{\beta}E|V\rangle, \\
\langle W|EE =& 0,\ \langle W|ED = p\langle W|D,\ \langle W|D|V\rangle = \frac{p}{\beta}.
\end{aligned} \quad (5)$$

These relations are closely related to those for the stationary state of the parallel-update TASEP with ordinary open boundary condition [11]. The normalization constant is expressed as

$$Z \;=\; 1 + \sum_{L\ge 1}\left(\frac{\alpha}{p(1-\alpha)}\right)^L \langle W|D(D+E)^{L-1}|V\rangle \quad (6)$$

$$\;=\; \frac{2(1-\alpha)\beta}{R-p+2(1-\alpha)\beta} \quad (7)$$

with $R = \sqrt{p(p-4\alpha(1-\alpha))}$. The average number of particles $\langle N\rangle$ and the average system length (the position of the leftmost particle) $\langle L\rangle$ are calculated as

$$\langle N\rangle = \frac{\alpha(1-\alpha)(p-2\alpha p+R)}{R(R-p+2(1-\alpha)\beta)}, \quad (8)$$

$$\langle L\rangle = \frac{\alpha p(R-p+2(1-\alpha))}{R(R-p+2(1-\alpha)\beta)} \quad (9)$$

in the stationary state. Note that $\langle N\rangle$ and $\langle L\rangle$ diverge on the critical line $\alpha = \frac{1}{2}(1-\sqrt{1-p})$, $\beta > 1-\sqrt{1-p}$, where the stationary state exists.

A generalization of the model, where the entry probability depends on the system length, also has a matrix product stationary state, see App. A.

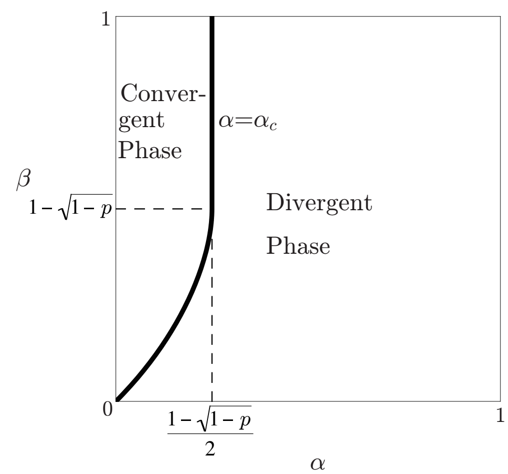

FIG. 2: Phase diagram for the EQP. The parameter space is divided into two regions with and without the stationary state.

III. DOMAIN WALL PICTURE AND MONTE CARLO SIMULATIONS

In this section, we discuss the time evolution of the average number of particles $\langle N_t\rangle$ and the average system length $\langle L_t\rangle$ corresponding to the position of the leftmost particle.

In the ordinary open boundary case, where the length of the system is fixed, a domain wall moves rightward or leftward, or exhibits a random walk depending on the boundary parameters [12]. In the same way, we will discuss how the system length $\langle L_t\rangle$ moves. We also observe how the average number of particles $\langle N_t\rangle$ changes as well. The continuity equation

$$\langle N_{t+1}\rangle - \langle N_t\rangle = J_t^{\rm in} - J_t^{\rm out}. \quad (10)$$

holds, where $J_t^{\rm in}$ and $J_t^{\rm out}$ are the flows of particles entering and leaving the system, respectively. The inflow $J_t^{\rm in}$ is always α, which is due to the fact that the site

where particles enter is by definition never blocked. In other words, our model is not a call-loss system. Under the assumption that the outflow $J_t^{\rm out}$ is independent of t, we have $\langle N_t\rangle = (\alpha - J^{\rm out})t + \langle N_0\rangle$. In fact our simulations show that both $\langle N_t\rangle$ and $\langle L_t\rangle$ decrease or increase linearly in time t according to $\alpha < \alpha_c$ or $\alpha > \alpha_c$, respectively.

A. Convergent Phase

When $\alpha < \alpha_c$, the system converges to the stationary state (3), (4). We impose the initial condition that particles are distributed uniformly with density

$$\rho = \begin{cases} \frac{1}{2} & \text{for } \beta > 1-\sqrt{1-p}, \\ \frac{p-\beta}{p-\beta^2} & \text{for } \beta \le 1-\sqrt{1-p}. \end{cases} \tag{11}$$

As in Fig. 3, $\langle N_t\rangle$ and $\langle L_t\rangle$ decrease linearly in time as

$$\langle N_t\rangle \sim (\alpha - J^{\rm out})t + \langle N_0\rangle, \tag{12}$$

$$\langle L_t\rangle \sim \frac{\alpha - J^{\rm out}}{\rho}t + \langle L_0\rangle, \tag{13}$$

with

$$\begin{aligned} J^{\rm out} &= \frac{1-\sqrt{1-4p\rho(1-\rho)}}{2} \\ &= \begin{cases} \frac{1-\sqrt{1-p}}{2} & \text{for } \beta > 1-\sqrt{1-p}, \\ \frac{\beta(p-\beta)}{p-\beta^2} & \text{for } \beta \le 1-\sqrt{1-p}. \end{cases} \end{aligned} \tag{14}$$

Since $J^{\rm out} = \alpha_c$, we have $\alpha - J^{\rm out} < 0$ which means that the domain wall moves rightward.

According to the forms for $J^{\rm out}$ and ρ, we call the phases

$$\text{MC-C:}\quad \alpha < \frac{1-\sqrt{1-p}}{2} \quad \text{and} \quad \beta > 1-\sqrt{1-p}, \tag{15}$$

$$\text{HD-C:}\quad \alpha < \frac{\beta(p-\beta)}{p-\beta^2} \quad \text{and} \quad \beta \le 1-\sqrt{1-p}, \tag{16}$$

maximal-current-convergent (MC-C) and high-density-convergent (HD-C) phases, respectively, see Fig. 4.

It should be noted that the outflow is given by Eqn. (14) only while $0 \le t \lesssim \frac{\langle L_0\rangle}{\alpha - J^{\rm out}}$. As $t \to \infty$, the outflow approaches α, assuring that $\langle N_t\rangle$ approaches the stationary value (8).

Figure 5 shows density profiles in the HD-C and MC-C phases. We can observe that the bulk density keeps its initial value (11).

B. Divergent Phase

When $\alpha > \alpha_c$, it is natural to expect that the domain wall moves leftward, and the time evolutions of $\langle N_t\rangle$ and $\langle L_t\rangle$ are expressed by Eqs. (12) and (13), respectively,

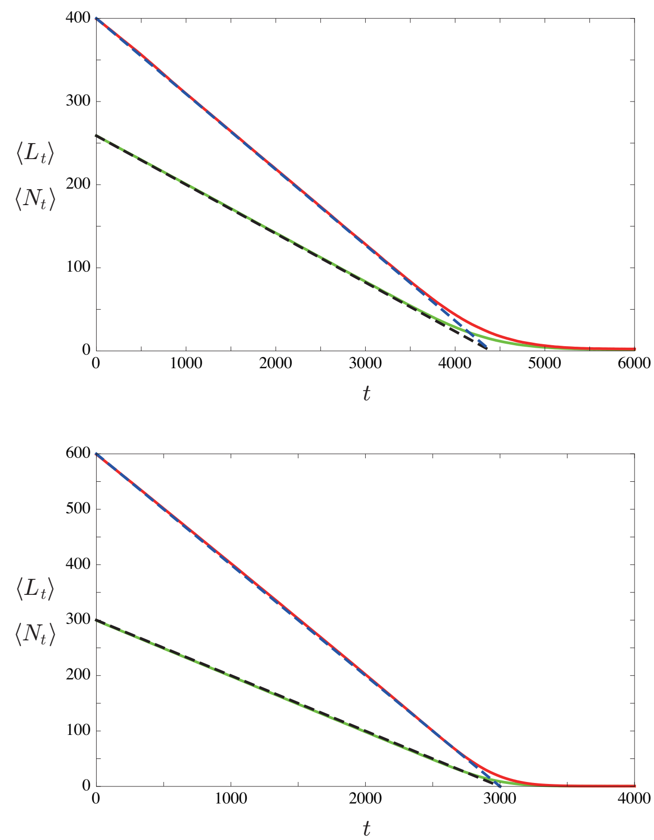

FIG. 3: Dynamics in the HD-C (top) and MC-C (bottom) phases. The parameter values are chosen as $(\alpha, \beta, p) = (0.2, 0.4, 0.84)$ and $(0.2, 0.8, 0.84)$, and the initial conditions as $(\langle N_0\rangle, \langle L_0\rangle) = (400\rho, 400)$ and $(600\rho, 600)$ (with ρ defined by (11)), respectively. The green and red lines are data for $\langle N_t\rangle$ and $\langle L_t\rangle$ obtained from Monte Carlo simulations, where 5000 samples are averaged. The black and blue lines correspond to the predictions of the domain wall theory. Note that the asymptotic values are small but non-zero (see Eqs. (8) and (9)): $(\langle N_\infty\rangle, \langle L_\infty\rangle) = (1.70, 2.28)$ and $(0.42, 0.56)$, respectively.

with the density (11) and the outflow (14). The simulations imply that this is true for $\langle N_t\rangle$ with

$$\langle N_t\rangle \sim (\alpha - J^{\rm out})t + \langle N_0\rangle, \tag{17}$$

but fails for $\langle L_t\rangle$, see Fig. 6. This failure is not unexpected since the predicted velocity $\frac{\alpha - J^{\rm out}}{\rho}$ can be greater than 1 whereas the length $\langle L_t\rangle$ can not be larger than $t + \langle L_0\rangle$ by the definition of the model. However, the simulation results (see Fig. 6) indicate that

$$\langle L_t\rangle \sim Vt + \langle L_0\rangle, \tag{18}$$

so that the prediction is qualitatively correct. The velocity V satisfies

$$V \to 0 \quad (\alpha \to \alpha_c), \tag{19}$$

$$V = 1 \quad (\alpha = 1). \tag{20}$$

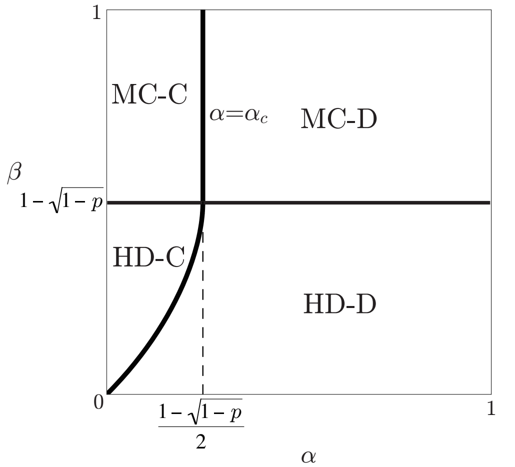

FIG. 4: Subphases of the EQP.

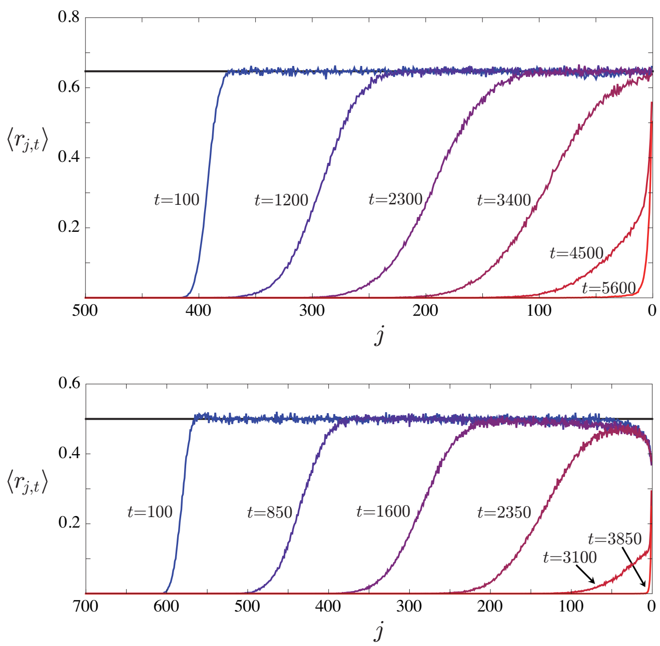

FIG. 5: Density profiles ($\langle r_{j,t}\rangle$ of the jth site at time t) in the HD-C (top) and MC-C (bottom) phases. The parameters (α, β, p) and the initial conditions $(\langle N_0\rangle, \langle L_0\rangle)$ are set to the same values as in Fig. 3. The colored snapshots are obtained by averaging 5000 samples of Monte Carlo simulations. The black lines represent the predicted densities ρ according to Eqn. (11).

and we have exactly $\langle L_t\rangle = t + \langle L_0\rangle$ when $\alpha = 1$. Moreover, when $p = 1$, we will show in the next section that

$$V = \alpha - \beta + \alpha\beta = \frac{\alpha - \frac{\beta}{1+\beta}}{\frac{1}{1+\beta}} = \frac{\alpha - J^{\rm out}}{\rho}. \tag{21}$$

Equations (17) and (18) can be regarded as the asymptotic behaviors

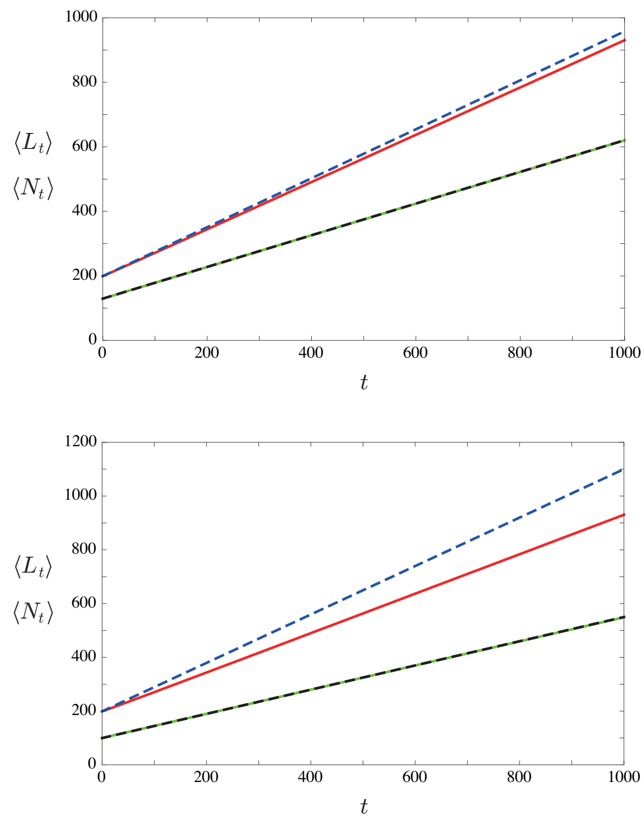

FIG. 6: Dynamics in the HD-D (top) and MC-D (bottom) phases. The parameter values are chosen as $(\alpha, \beta, p) = (0.75, 0.4, 0.84)$ and $(0.75, 0.8, 0.84)$, respectively, and the initial conditions as $(\langle N_0\rangle, \langle L_0\rangle) = (200\rho, 200)$ (with ρ defined by (11)). The green and red lines are data for $\langle N_t\rangle$ and $\langle L_t\rangle$ obtained from Monte Carlo simulations, where 1000 samples are averaged. The black and blue lines correspond to the predictions of the domain wall theory.

$$\langle N_t\rangle = (\alpha - J^{\rm out})t + o(t), \tag{22}$$
$$\langle L_t\rangle = Vt + o(t). \tag{23}$$

In the same way as in the convergent phase, we call the subphases

$$\text{MC-D:}\quad \alpha > \frac{1-\sqrt{1-p}}{2} \quad \text{and} \quad \beta > 1-\sqrt{1-p}, \tag{24}$$
$$\text{HD-D:}\quad \alpha > \frac{\beta(p-\beta)}{p-\beta^2} \quad \text{and} \quad \beta \le 1-\sqrt{1-p}, \tag{25}$$

maximal-current-divergent (MC-D) and high-density-divergent (HD-D) phases, respectively. Note that the densities in the MC-D and HD-D phases are higher than (or equal to) $\frac{1}{2}$ and $\frac{p-\beta}{p-\beta^2}$, respectively.

It is difficult to predict how $\langle N_t\rangle$ or $\langle L_t\rangle$ behaves just on the critical line $\alpha = \alpha_c$. For $p = 1$, however, we will

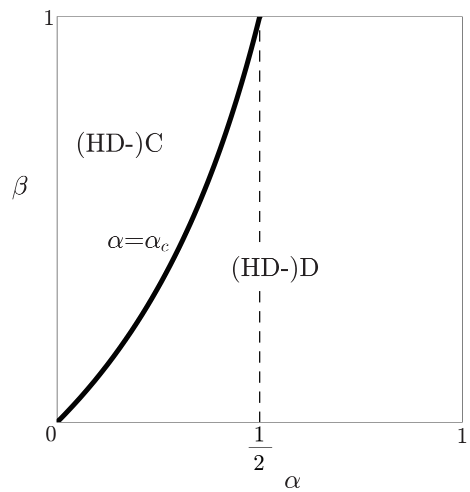

FIG. 7: Phase diagram for $p = 1$.

find in the next section diffusive behavior on the critical line as

$$\langle N_t \rangle = D_N \sqrt{t} + o(\sqrt{t}), \tag{26}$$

$$\langle L_t \rangle = D_L \sqrt{t} + o(\sqrt{t}) \tag{27}$$

with constants D_N and D_L.

IV. ASYMPTOTIC BEHAVIORS FOR $p = 1$

In this section we investigate the asymptotic behaviors of $\langle N_t \rangle$ and $\langle L_t \rangle$ rigorously for $p = 1$. Thanks to the deterministic particle hopping we can obtain the generating functions of $\langle N_t \rangle$ and $\langle L_t \rangle$. For simplicity, we impose the initial condition $\emptyset$ (there is no particle in the system at time $t = 0$), and reset the state space as

$$\begin{aligned}\widetilde{S} &:= \{\emptyset, 1, 10, 11, 101, 110, 111, 1010, 1011, \dots\} \\ &= \{\tau \in S | \tau \text{ does not contain sequence } 00\}.\end{aligned} \tag{28}$$

Note that, for $p = 1$, the sequence 00 never appears if the system starts from the initial condition $\emptyset$. In this case, the MC-D and MC-C phases vanish from the phase diagram, see Fig. 7.

We first consider the number of particles, borrowing the classification from [9] as

$$P_t^A(N) = \text{Prob}\left[\begin{array}{l}\#\text{ of particles is } N \text{ at time } t \\ \wedge \text{ site 1 is occupied at time } t\end{array}\right], \tag{29}$$

$$P_t^B(N) = \text{Prob}\left[\begin{array}{l}\#\text{ of particles is } N \text{ at time } t \\ \wedge \text{ site 1 is empty at time } t\end{array}\right] \tag{30}$$

for $N \in \mathbb{Z}_{\geq 0}$ with $P_t^A(0) \equiv 0$. These probabilities are governed by the following master equation, which was found in [9]:

$$\begin{aligned}P_{t+1}^A(1) =& (1-\alpha)(1-\beta)P_t^A(1) + \alpha P_t^B(0) \\ &+ (1-\alpha)P_t^B(1),\end{aligned} \tag{31}$$

$$\begin{aligned}P_{t+1}^A(N) =& \alpha(1-\beta)P_t^A(N-1) + \alpha P_t^B(N-1) \\ &+ (1-\alpha)(1-\beta)P_t^A(N) + (1-\alpha)P_t^B(N),\end{aligned} \tag{32}$$

$$P_{t+1}^B(0) = (1-\alpha)P_t^B(0) + (1-\alpha)\beta P_t^A(1), \tag{33}$$

$$P_{t+1}^B(N) = \alpha\beta P_t^A(N) + (1-\alpha)\beta P_t^A(N+1). \tag{34}$$

This simplification is due to the deterministic hopping $p = 1$. We choose the initial condition such that

$$P_0^B(0) = 1, \quad P_0^A(N) = P_0^B(N) = 0 \quad (N \in \mathbb{N}). \tag{35}$$

We will check that the average number of particles $\langle N_t \rangle$ converges to the stationary value (8) when $\alpha < \frac{\beta}{1+\beta}$, and show that $\langle N_t \rangle$ behaves as Eqn. (22) when $\alpha > \frac{\beta}{1+\beta}$. We will also show that $\langle N_t \rangle$ exhibits diffusive behavior on the critical line $\alpha = \frac{\beta}{1+\beta}$.

We define the generating functions of $P_t^A(N)$ and $P_t^B(N)$ as

$$G_z^A(N) = \sum_{t \geq 0} P_t^A(N) z^t, \tag{36}$$

$$G_z^B(N) = \sum_{t \geq 0} P_t^B(N) z^t, \tag{37}$$

for $|z| < 1$. Noting the initial condition (35), we find

$$\begin{aligned}G_z^A(1) =& (1-\alpha)(1-\beta)zG_z^A(1) + \alpha z G_z^B(0) \\ &+ (1-\alpha)zG_z^B(1),\end{aligned} \tag{38}$$

$$\begin{aligned}G_z^A(N) =& \alpha(1-\beta)zG_z^A(N-1) + \alpha z G_z^B(N-1) \\ &+ (1-\alpha)(1-\beta)zG_z^A(N) + (1-\alpha)zG_z^B(N),\end{aligned} \tag{39}$$

$$G_z^B(0) - 1 = (1-\alpha)zG_z^B(0) + (1-\alpha)\beta z G_z^A(1), \tag{40}$$

$$G_z^B(N) = \alpha\beta z G_z^A(N) + (1-\alpha)\beta z G_z^A(N+1). \tag{41}$$

From Eqn. (40), we have

$$G_z^B(0) = \frac{1 + (1-\alpha)\beta z G_z^A(1)}{1 - (1-\alpha)z}. \tag{42}$$

Inserting this and Eqn. (41) into Eqs. (38) and (39), we get a recurrence formula for $G_z^A(N)$ as

$$G_z^A(2) = -\frac{\alpha}{(1-\alpha)^2\beta(1-(1-\alpha)z)z} + \frac{1-(1-\alpha)(2-\beta)z+(1-\alpha)(1-\alpha-\beta-\alpha\beta)z^2+(1-\alpha)^2\alpha\beta z^3}{(1-\alpha)^2\beta(1-(1-\alpha)z)z^2} G_z^A(1) \tag{43}$$

$$G_z^A(N+1) = -\frac{1-(1-\alpha)(1-\beta)z-2(1-\alpha)\alpha\beta z^2}{(1-\alpha)^2\beta z^2} G_z^A(N) + \frac{\alpha(1-\beta)+\alpha^2\beta z}{(1-\alpha)^2\beta z} G_z^A(N-1)$$
$$\left(=: xG_z^A(N) + yG_z^A(N-1) \right) \qquad (N \in \mathbb{Z}_{\geq 2}). \tag{44}$$

The recurrence formula (44) has the following solution:

$$\begin{aligned} G_z^A(N) =& \frac{\lambda_+^{N-1}}{\lambda_+ - \lambda_-} \left(G_z^A(2) - \lambda_- G_z^A(1) \right) \\ &- \frac{\lambda_-^{N-1}}{\lambda_+ - \lambda_-} \left(G_z^A(2) - \lambda_+ G_z^A(1) \right), \end{aligned} \tag{45}$$

where

$$\lambda_\pm = \frac{1-(1-\alpha)(1-\beta)z-2(1-\alpha)\alpha\beta z^2 \pm r}{2(1-\alpha)^2\beta z^2} \tag{46}$$

with

$$r = \sqrt{(1-(1-\alpha)(1-\beta)z)^2 - 4(1-\alpha)\alpha\beta z^2} \tag{47}$$

are solutions to $\lambda^2 = x\lambda + y$ with x and y as defined in Eqn. (44). Restricting the "initial condition" $G_z^A(1)$ and $G_z^A(2)$ such that

$$\left| G_z^A(N) \right| < \sum_{t\geq 0} |z|^t = \frac{1}{1-|z|}, \tag{48}$$

we have

$$G_z^A(2) - \lambda_- G_z^A(1) = 0. \tag{49}$$

(Note that $0 < |\lambda_-| < 1 < |\lambda_+|$.) From this constraint and the relation (43) $G_Z^A(1)$ is determined, and we find

$$G_z^A(N) = \lambda_-^{N-1} G_z^A(1) = \lambda_-^{N-1} \frac{-1+(1-\alpha)(2-\beta)z-(1-\alpha)(1-\alpha-\beta-\alpha\beta)z^2+(1-(1-\alpha)z)r}{2(1-\alpha)^2\beta^2(1-z)z^2} \quad (N \in \mathbb{N}), \tag{50}$$

$$G_z^B(0) = \frac{-1+r+(1-\alpha)(1+\beta)z}{2(1-\alpha)\beta z(1-z)}, \qquad G_z^B(N) = (\alpha + (1-\alpha)\lambda_-)\, \beta z \lambda_-^{N-1} G_z^A(1) \quad (N \in \mathbb{N}). \tag{51}$$

Then we obtain the generating function $G_z(N)$ of the probability that the number of particles is N as

$$G_z(0) = G_z^B(0), \quad G_z(N) = G_z^A(N) + G_z^B(N) = g(z)\lambda_-^N, \tag{52}$$

$$g(z) = \frac{\beta - 1 + \left(1-\alpha-\beta+\beta^2-\alpha\beta^2\right)z + (1-\alpha)(1-\beta)\beta z^2 + r(1-(1-z)\beta)}{2\beta(1-\alpha)(1-(1-z\alpha)\beta)(1-z)z} \quad (N \in \mathbb{N}). \tag{53}$$

We also introduce the generating function $\mathcal{G}_{z\zeta}$ of the generating function $G_z(N)$ as

$$\begin{aligned} \mathcal{G}_{z\zeta} &= \sum_{N\geq 0} G_N(z)\zeta^N = G_0(z) + \sum_{N\geq 1} g(z)(\lambda_-\zeta)^N \\ &= G_0(z) + g(z)\frac{\lambda_-\zeta}{1-\lambda_-\zeta}, \end{aligned} \tag{54}$$

and K_z of the average number of particles as

$$\begin{aligned} K_z &= \sum_{t\geq 0} z^t \langle N_t \rangle = \sum_{t\geq 0} z^t \sum_{N\geq 0} N \left(P_t^A(N) + P_t^B(N) \right) \\ &= \frac{\partial}{\partial\zeta}\mathcal{G}_{z\zeta}\bigg|_{\zeta=1} = \frac{z(2\alpha-1+z(1-\alpha)(1-\beta)+r)}{2(1-z)^2(1+z\beta)}. \end{aligned} \tag{55}$$

The asymptotic behavior of $\langle N_t \rangle$ is determined by the degree of the singularity $z = 1$ of K_z [13]. When $\alpha <$

$\frac{\beta}{1+\beta}$, we find

$$(1-z)K_z\Big|_{z\to 1} = \frac{\alpha(1-\alpha)}{\beta-\alpha-\alpha\beta}, \tag{56}$$

and $\langle N_t\rangle$ converges as

$$\langle N_t\rangle \to \frac{\alpha(1-\alpha)}{\beta-\alpha-\alpha\beta} \qquad (t\to\infty). \tag{57}$$

Of course this limit value agrees with the stationary value (8) with $p=1$. When $\alpha > \frac{\beta}{1+\beta}$, we find

$$(1-z)^2K_z\Big|_{z\to 1} = \alpha - \frac{\beta}{1+\beta}, \tag{58}$$

and $\langle N_t\rangle$ behaves as

$$\langle N_t\rangle = \left(\alpha - \frac{\beta}{1+\beta}\right)t + o(t) \tag{59}$$

$$= (\alpha - J^{\mathrm{out}})t + o(t) \qquad (t\to\infty). \tag{60}$$

When $\alpha = \frac{\beta}{1+\beta}$, we find

$$(1-z)^{\frac{3}{2}}K_z\Big|_{z\to 1} = \sqrt{\frac{\beta}{(1+\beta)^3}}, \tag{61}$$

and $\langle N_t\rangle$ behaves as

$$\langle N_t\rangle = 2\sqrt{\frac{\beta t}{\pi(1+\beta)^3}} + o\left(\sqrt{t}\right) \qquad (t\to\infty). \tag{62}$$

Now we turn to the behavior of the length of the system (the position of the leftmost particle). Let $Q_t(L)$ be the probability that the system length is L at time t. The probability $Q_t(L)$ is governed by

$$Q_{t+1}(0) = (1-\alpha)Q_t(0) + \beta(1-\alpha)Q_t(1), \tag{63}$$

$$Q_{t+1}(L) = \alpha Q_t(L-1) + (1-\alpha)(1-\beta)Q_t(L) + (1-\alpha)\beta Q_t(L+1) \tag{64}$$

for $L\in\mathbb{N}$. The first equation means that, if there is no particle at time $t+1$, there is no particle at time t and no particle enters (with probability $1-\alpha$), or there is only one particle on the rightmost site at time t which leaves the system and no particle enters (with probability $(1-\alpha)\beta$). The second equation is derived in App. B.

In the same way as for the number of particles, we define the generating function $M_z(L) = \sum_{t\ge 0} z^t Q_t(L)$ ($|z|<1$). Noting the initial condition $Q_0(0)=1$ and $Q_0(L)=0$ ($L\in\mathbb{N}$), we find

$$M_z(1) = \frac{1-(1-\alpha)z}{(1-\alpha)\beta z}M_z(0) - \frac{1}{(1-\alpha)\beta z}, \tag{65}$$

$$\begin{aligned} M_z(L+1) = {} & \frac{1-(1-\alpha)(1-\beta)z}{(1-\alpha)\beta z}M_z(L) \\ & - \frac{\alpha}{(1-\alpha)\beta}M_z(L-1) \\ & \big(=: XM_z(L) + YM_z(L-1)\big). \end{aligned} \tag{66}$$

The solution to the recurrence formula (66) is

$$\begin{aligned} M_z(L) = {} & \frac{\Lambda_+^L}{\Lambda_+-\Lambda_-}\left(M_z(1) - \Lambda_- M_z(0)\right) \\ & - \frac{\Lambda_-^L}{\Lambda_+-\Lambda_-}\left(M_z(1) - \Lambda_+ M_z(0)\right), \end{aligned} \tag{67}$$

where $\Lambda_\pm = \frac{1-(1-\alpha)(1-\beta)z\pm r}{2(1-\alpha)\beta z}$ are the solutions to $\Lambda^2 = X\Lambda + Y$ with X and Y as defined in Eqn. (66). Due to the condition $|M_z(L)| < \frac{1}{1-|z|}$, the "initial condition" must be restricted as

$$M_z(1) - \Lambda_- M_z(0) = 0. \tag{68}$$

(Note that $0<|\Lambda_-|<1<|\Lambda_+|$.) Thus we find

$$M_z(L) = M_z(0)\Lambda_-^L = \frac{2}{1-(1-\alpha)(1+\beta)z+r}\Lambda_-^L. \tag{69}$$

The generating function of the generating function is calculated as

$$\mathcal{M}_{z\xi} = \sum_{L\ge 0}\xi^L M_z(L) = \frac{M_z(0)}{1-\xi\Lambda_-}, \tag{70}$$

and that of the average system length as

$$\begin{aligned} S_z &= \sum_{t\ge 0} z^t\langle L_t\rangle = \sum_{t\ge 0} z^t \sum_{L\ge 0} LQ_t(L) \\ &= \frac{\partial}{\partial\xi}\mathcal{M}_{z\xi}\Big|_{\xi=1} = \frac{-1+(1+\alpha-\beta+\alpha\beta)z+r}{2(1-z)^2}. \end{aligned} \tag{71}$$

The asymptotic behavior of $\langle L_t\rangle$ is determined by the degree of the singularity $z=1$ of S_z. When $\alpha<\frac{\beta}{1+\beta}$, we find

$$(1-z)S_z\Big|_{z\to 1} = \frac{\alpha}{\beta-\alpha-\alpha\beta}, \tag{72}$$

and $\langle L_t\rangle$ converges as

$$\langle L_t\rangle \to \frac{\alpha}{\beta-\alpha-\alpha\beta} \qquad (t\to\infty). \tag{73}$$

Again this limit value agrees with the stationary value (9) with $p=1$. When $\alpha > \frac{\beta}{1+\beta}$, we find

$$(1-z)^2S_z\Big|_{z\to 1} = \alpha-\beta+\alpha\beta, \tag{74}$$

and $\langle L_t\rangle$ behaves as

$$\langle L_t\rangle = \frac{\alpha - \frac{\beta}{1+\beta}}{\frac{1}{1+\beta}}t + o(t) \tag{75}$$

$$= \frac{\alpha - J^{\mathrm{out}}}{\rho}t + o(t) \qquad (t\to\infty). \tag{76}$$

When $\alpha = \frac{\beta}{1+\beta}$, we find

$$(1-z)^{\frac{3}{2}}S_z\Big|_{z\to 1} = \sqrt{\frac{\beta}{1+\beta}}, \tag{77}$$

and $\langle L_t \rangle$ behaves as

$$\langle L_t \rangle = 2\sqrt{\frac{\beta t}{\pi(1+\beta)}} + o\left(\sqrt{t}\right) \qquad (t \to \infty). \tag{78}$$

V. CONCLUSION

We have investigated the dynamical properties of the EQP, a queueing process with excluded-volume effect. The model can be interpreted as a TASEP with varying length. Using generating function techniques and a phenomenological domain wall theory we have derived analytical predictions for the time-dependence of the number of particles $\langle N_t \rangle$ and the average system length $\langle L_t \rangle$.

We found that the two phases observed previously can be divided in subphases. The convergent phase, where the system length remains finite, consists of high-density and maximal current subphases. The same is true for the diverging phase, where the system length becomes infinite in the long time limit.

By comparing with Monte Carlo simulations it was found that the predications of the domain wall theory for the dynamical behavior are at least qualitatively correct, i.e. $\langle L_t \rangle$ and $\langle N_t \rangle$ converge or diverge linearly in time. Moreover they are in good agreement even quantitively in the convergent phase. In the divergent phase, the predicted velocity for $\langle N_t \rangle$ appears to be correct whereas deviations from the domain wall theory can be observed for $\langle L_t \rangle$.

For $p = 1$, we derived exact analytical results for the behaviors of $\langle N_t \rangle$ and $\langle L_t \rangle$ by using the generating function method. We showed the linearity of their dynamics in the divergent phase as predicted by the domain wall theory. We found diffusive behavior on the critical line as well.

The simple approach presented here does not provide a good expression for the velocity V of the system length ($\langle L_t \rangle \sim Vt$) in the divergent phase. Here further analysis of the detailed density profiles in the divergent phase may be helpful. A first step would be the numerical determination of V to get a better understanding of its dependence on the parameters α, β and p. Another way to settle the problem is extending the exact result to $p < 1$ case, which may be difficult but very worthwhile.

Appendix A: Stationary State for Inhomogeneous Injection Case

Here we consider the stationary state for a generalized model where the entry probability depends on the system length. A new particle enters the system with probability α_L if the leftmost occupied site is L, or α_0 if there is no particle on the chain. The stationary state of this generalized model can be written in the following matrix product form with the same matrices and vectors

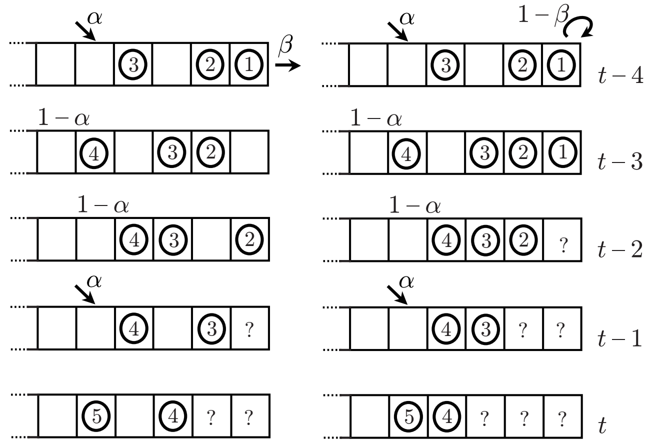

FIG. 8: Example for transitions from 1011 at time $t-4$ to the states in which the system length is 5 and the 4th site is empty (left) or occupied (right) at time t.

($D, E, \langle W|$ and $|V\rangle$):

$$P(\tau_L \cdots \tau_1) = \frac{1}{Z} \frac{\prod_{j=0}^{L-1} \alpha_j}{p^L \prod_{j=1}^{L}(1-\alpha_j)} \langle W| X_{\tau_L} \cdots X_{\tau_1} |V\rangle, \tag{A1}$$

$$Z = 1 + \sum_{L \geq 1} \frac{\prod_{j=0}^{L-1} \alpha_j}{p^L \prod_{j=1}^{L}(1-\alpha_j)} \langle W| D(D+E)^{L-1} |V\rangle. \tag{A2}$$

This can be proved in the same way as in the homogeneous case $\alpha_L = \alpha$, see [1].

Appendix B: Derivation of Eqn. (64)

To derive Eqn. (64), we show

$$\beta Q_t(L) = \mathrm{Prob}\left[\begin{array}{l}\text{the system length is } L \text{ at time } t \\ \wedge\ (L-1)\text{th site is empty at time } t\end{array}\right], \tag{B1}$$

$$(1-\beta) Q_t(L) = \mathrm{Prob}\left[\begin{array}{l}\text{the system length is } L \text{ at time } t \\ \wedge\ (L-1)\text{th site is occupied at time } t\end{array}\right], \tag{B2}$$

for $L \in \mathbb{N}_{\geq 2}$

In the deterministic hopping case $p = 1$ with the initial condition $\emptyset$, the sequence 00 (except the infinite number of 0's left to the leftmost particle) never appears, and holes necessarily "hop" leftward. Thus the first site must be occupied by a particle at time $t - L + 1$ if the system length at time t is L. The local state (empty or occupied) of the $(L-1)$th site at time t depends only on whether the first particle exits or not at time $t - L + 1$. Let the number of particles be n at time $t - L + 1$, and label the

particles by natural numbers as in Fig. 8. We can show by induction that the ℓth particle is on the ℓth site at time $t-L+\ell$. Thus we find that $L-n$ new particles should enter the system during $[t-L+1, t-1]$.

We obtain (the summation of) the transition probability from a configuration $S^{(t-L+1)}$ at time $t-L+1$ to the states in which the system length is L and the $(L-1)$th site is empty or occupied at time t is given by

$$\sum_{\substack{S^{(t)},\dots,S^{(t-L+2)}\in\widetilde{S} \\ |S^{(t)}|=L,(S^{(t)})_{L-1}=0}} \prod_{t'=t-L+1}^{t-1} W(S^{(t')}\to S^{(t'+1)}) = \begin{pmatrix} L-1 \\ L-n \end{pmatrix} \alpha^{L-n}(1-\alpha)^{n-1}\beta, \tag{B3}$$

$$\sum_{\substack{S^{(t)},\dots,S^{(t-L+2)}\in\widetilde{S} \\ |S^{(t)}|=L,(S^{(t)})_{L-1}=1}} \prod_{t'=t-L+1}^{t-1} W(S^{(t')}\to S^{(t'+1)}) = \begin{pmatrix} L-1 \\ L-n \end{pmatrix} \alpha^{L-n}(1-\alpha)^{n-1}(1-\beta), \tag{B4}$$

respectively, where $W(S^{(t')}\to S^{(t'+1)})$, $|S^{(t')}|$ and $(S^{(t')})_j$ denote the transition probability from $S^{(t')}$ to $S^{(t'+1)}$, the length of the state $S^{(t')}$ and the local state of site j. The binomial $\begin{pmatrix} L-1 \\ L-n \end{pmatrix}$ gives the number of possibilities for when the new particles enter the system. These equations lead to Eqs. (B1) and (B2).

If the length is L at time $t+1$, there are the following three possibilities at time t:

(i) the length is $L-1$ and a new particle enters (with probability α),

(ii) the length is L, the $(L-1)$th site is occupied, and no particle enters (with probability $1-\alpha$),

(iii) the length is $L+1$, the Lth site is empty, and no particle enters (with probability $1-\alpha$).

Then we achieve Eqn. (64) for $L\in\mathbb{Z}_{\geq 2}$. For $L=1$, the case (ii) is replaced by

> the length is 1, the particle at the rightmost site does not leave, and no particle enters (with probability $(1-\alpha)(1-\beta)$).

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